Log Rule for Integration

The differentiation rules

$$\frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x} \qquad \text{and} \qquad \frac{d}{dx}\left[\ln|u|\right] = \frac{u'}{u}$$

Theorem: Log Rule for Integration

Let u be a differentiable function of x.

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \frac{1}{u}\,du = \ln|u| + C$

Because $du = u'\,dx$, the second formula can also be written as

$$\int \frac{u'}{u}\,du = \ln|u| + C$$

Example: Using the Log Rule for Integration

$$\int \frac{2}{x}\,dx = 2\int \frac{1}{x}\,dx$$

$$= 2\ln|x| + C$$

$$= \ln(x^2) + C$$

Because x^2 cannot be negative, the absolute value is ..unnecessary in the final form of the antiderivative

Example: Using the Log Rule with a Change of Variable

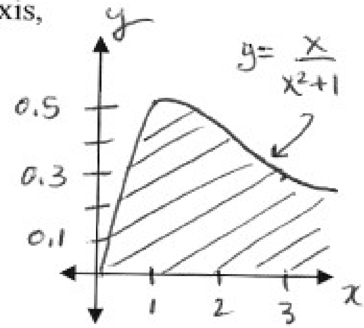
Evaluate $\int \frac{1}{4x-1}\,dx$

Solution: Let $u = 4x-1$
Then $du = 4dx$

$$\int \frac{1}{4x-1}\,dx = \frac{1}{4}\int\left(\frac{1}{4x-1}\right)4dx$$
$$= \frac{1}{4}\int \frac{1}{u}\,du$$
$$= \frac{1}{4}\ln|u| + C$$
$$= \frac{1}{4}\ln|4x-1| + C$$

Example: Finding the Area with the Log Rule

Find the area of the region bounded the graph of $y = \frac{x}{x^2+1}$, the x-axis, and the vertical lines $x = 3$



Solution:

By visual inspection you can see the area is bound by $x=0$ and $x=3$.

Use the definite integral:

$$\int_0^3 \frac{x}{x^2+1}\,dx$$

Let $u = x^2+1$
$du = 2x\,dx$

$$= \frac{1}{2}\int_0^3 \frac{2x}{x^2+1}\,dx = \frac{1}{2}\int_0^3 \frac{1}{u}\,du = \frac{1}{2}\Big[\ln|u|\Big]_0^3$$
$$= \frac{1}{2}\Big[\ln|x^2+1|\Big]_0^3$$
$$= \frac{1}{2}(\ln 10 - \ln 1) = \frac{1}{2}\ln 10 \approx 1.151$$

Example:

Let $u = x^3 + x$, Then $du = (3x^2+1)dx$

a. $\displaystyle\int \frac{3x^2+1}{x^3+x}\,dx = \ln|x^3+x| + C$

Let $u = \tan x$, Then $du = \sec^2 x\,dx$

b. $\displaystyle\int \frac{\sec^2 x}{\tan x}\,dx = \ln|\tan x| + C$

Let $u = x^2+2x$, Then $du = (2x+2)dx$

c. $\displaystyle\int \frac{x+1}{x^2+2x}\,dx = \frac{1}{2}\int \frac{2(x+1)}{x^2+2x}\,dx = \frac{1}{2}\ln|x^2+2x| + C$

Let $u = 3x+2$, Then $du = 3dx$

d. $\displaystyle\int \frac{1}{3x+2}\,dx = \frac{1}{3}\int \frac{3}{3x+2}\,dx = \frac{1}{3}\ln|3x+2| + C$

Example: Using the Long Division before Integrating

Evaluate $\displaystyle\int \frac{x^2+x+1}{x^2+1}\,dx$

<u>Solution:</u> Begin by using long division to rewrite the integrand

$$\frac{x^2+x+1}{x^2+1} \Rightarrow \begin{array}{r|l} & \quad 1 \\ x^2+1 & x^2+x+1 \\ & -(x^2+\cancel{1}\quad 1) \\ \hline & \quad x \end{array} \Rightarrow 1 + \frac{x}{x^2+1}$$

Now, integrate:

$$\int \frac{x^2+x+1}{x^2+1}\,dx = \int\left(1+\frac{x}{x^2+1}\right)dx = \int dx + \int \frac{x}{x^2+1}\,dx$$

$$= x + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = x + \frac{1}{2}\ln(x^2+1) + C$$

Example: Change of Variables with the Log Rule

Evaluate $\int \frac{2x}{(x+1)^2}\,dx$

notice

Solution: Let $u = x+1$, Then $du = dx$ and $x = u-1$

$$= \int \frac{2(u-1)}{u^2}\,du = 2\int \left(\frac{u}{u^2} - \frac{1}{u^2}\right)du$$

$$= 2\int \frac{du}{u} - 2\int u^{-2}\,du$$

$$= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C$$

$$= 2\ln|u| + \frac{2}{u} + C$$

$$= 2\ln|x+1| + \frac{2}{x+1} + C$$

HW p.327, #1-8, #33, #34