The Differential Equation

Let's solve more general types of differential equations.

Separation of Variables = Strategy used to rewrite equation so that each variable occurs on only one side of equation.

Example:

Solve the differential equation $y' = 2x/y$ and the x-axis $(0 \le x \le \pi)$ about the x-axis

Solution:

$$y' = \frac{2x}{y}$$

$$yy' = 2x$$

$$\int yy'\,dx = \int 2x\,dx$$

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 = x^2 + C_1$$

$$y^2 - 2x^2 = C$$

check by implicit diff.

$$\frac{dy}{dx} = y'$$

$$dy = y'\,dx$$

Note: consider Leibniz notation:

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$\int y\,dy = \int 2x\,dx$$

Growth and Decay Models

Rate of change of y — is — proportional to y.

$$\frac{dy}{dx} = ky$$

Theorem: Exponential Growth and Decay Model

If y is a differential function of t such that $y > 0$ and $y' = ky$, for some constant k, then

$$y = Ce^{kt}$$

C is the initial value of y, and k is the proportionality constant. Exponential growth occurs when $k > 0$, and exponential decay occurs when $k < 0$.

Example: Using and Exponential Growth Model

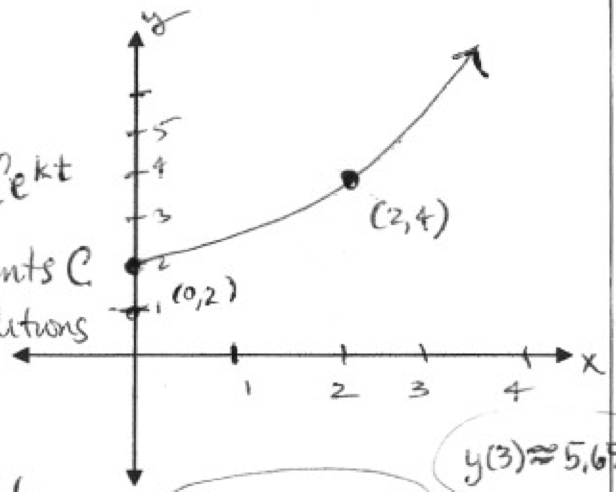
The rate of change of y is proportional to y. When $t = 0$, $y = 2$. When $t = 2$, $y = 4$. What is the value of y when $t = 3$?

Solution: Because $y' = ky$

we know that:

y and t are related by: $y = Ce^{kt}$

Let's find the values of the constants C and k by applying initial conditions



$$2 = Ce^{0} \Rightarrow C = 2$$

$$4 = 2e^{2k} \Rightarrow k = \tfrac{1}{2}\ln 2 \approx 0.3466$$

$y(3) \approx 5.657$

Therefore → $y = 2e^{0.3466t}$

Example: Radioactive Decay

Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram.?

Given: Plutonium half-life = 24, 360 years

Solution: Note: Rate of decay is proportional to mass

y = mass

Thus, we know $y = Ce^{kt}$

To find the values of C and k, apply the initial conditions:

$y = 10$ when $t = 0$

$$10 = Ce^{k(0)} = Ce^{0} = C$$

Next, use the fact that: $y = 5$ when $t = 24,360$

$$5 = 10e^{k(24,360)}$$

$$k = \frac{1}{24,360}\ln\left(\tfrac{1}{2}\right) \approx -2.8454 \times 10^{-5}$$

Therefore:

$$y = 10e^{-.000028454t}$$

$t \approx 80,923$ yrs ← $y = 1 = 10e^{-.000028454t}$

Example: Population Growth

Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

<u>Solution:</u> Let $y = Ce^{kt}$

$y = 100$ when $t = 2$
$y = 300$ when $t = 4$

Thus, $100 = Ce^{2k}$ and $300 = Ce^{4k}$

Use the first equation to "isolate C":

$$C = 100e^{-2k}$$

Substitute into second equation:

$$300 = 100e^{-2k}e^{4k} = 100e^{2k}$$
$$3 = e^{2k}$$
$$k = \tfrac{1}{2}\ln 3 \approx 0.5493$$

Therefore $y = Ce^{.5493t}$

solve for C with $y = 100$, $t = 2$

$$100 = Ce^{.5493(2)}$$
$$C \approx 33$$

Example: Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. If the sales follow an exponential pattern of decline, what will they be after another 2 months?

<u>Solution:</u> Use exponential decay model $y = Ce^{kt}$

First, $\rightarrow$ $C = 100{,}000$ when $t = 0$

because

$$100{,}000 = Ce^{k(0)}$$
$$100{,}000 = C$$

Next, $y = 80{,}000$ when $t = 4$

$$80{,}000 = 100{,}000e^{k(2)}$$
$$k = \tfrac{1}{4}\ln(0.8) \approx -0.0558$$

after 2 more months ($t = 6$)

$$y \approx 100{,}000\,e^{-0.0558(6)} \approx 71{,}500 \text{ units}$$

HW p. 363-364 #1-6:all; #17-20:all; #21-25:odd; #39-41:odd