Calculus
Lecture Notes: (7.7) Indeterminate Forms and L'Hopital's Rule
Marshall Math

Indeterminate Forms

Indeterminate forms: $\frac{0}{0}$ and $\frac{\infty}{\infty}$

L'Hopital's Rule

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

This works for all of the listed conditions:

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad \frac{-\infty}{\infty},\quad \frac{\infty}{-\infty},\quad \frac{-\infty}{-\infty}$$

Example:

Evaluate $\lim\limits_{x\to 0}\frac{e^{2x}-1}{x}$

Solution: Because direct substitution results in the indeterminate form $\frac{0}{0}$

you can apply L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{e^{2x-1}}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}[e^{2x}-1]}{\frac{d}{dx}[x]}=\lim_{x\to 0}\frac{2e^{2x}-0}{1}=\frac{2}{1}=2$$

Another form of L'Hopital's Rule states:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

Provided that : the limit on the right exists.

Example: ~~Apply the Rule Twice~~

Evaluate $\lim_{x\to\infty} \frac{\ln x}{x}$

Solution: Because direct substitution results in the indeterminate form $\frac{\infty}{\infty}$, you can apply L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

Example: Occasionally, you'll need to apply the rule twice

Evaluate $\lim_{x\to -\infty} \frac{x^2}{e^{-x}}$

Solution: Because direct substitution results in the indeterminate form $\frac{\infty}{\infty}$, you can apply L'Hôpital's Rule:

$$\lim_{x\to -\infty} \frac{x^2}{e^{-x}} = \lim_{x\to -\infty} \frac{2x}{-e^{-x}}$$ ⟨This yields $\frac{-\infty}{-\infty}$, ∴ apply L'Hôpital's Rule again⟩

$$\lim_{x\to -\infty} \frac{2x}{-e^{-x}} = \lim_{x\to -\infty} \frac{2}{e^{-x}} = 0$$

There are other indeterminate forms: such as $0\cdot\infty$, 1^{∞}, ∞^0, 0^0, and $\infty-\infty$

$\underbrace{\lim_{x\to 0}(x)\left(\frac{1}{x}\right)}_{\text{limit}=1}$ $\quad$ $\underbrace{\lim_{x\to 0}(x)\left(\frac{2}{x}\right)}_{\text{limit}=2}$ $\quad$ $\underbrace{\lim_{x\to \infty}(x)\left(\frac{1}{e^x}\right)}_{\text{limit}=0}$ $\quad$ $\underbrace{\lim_{x\to \infty}(e^x)\left(\frac{1}{x}\right)}_{\text{limit}=\infty}$

These limits may require techniques to convert into forms: ~~Because each limit is different, ...~~ $\frac{0}{0}$ and $\frac{\infty}{\infty}$ so you can use L'Hôpital's

Example:

Evaluate $\lim_{x\to\infty} e^{-x}\sqrt{x}$ Direct substitution yields: $0\cdot\infty$, we need to Rewrite to fit the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Solution:

$$\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty}\frac{\sqrt{x}}{e^x} = \frac{\infty}{\infty}$$ (This we can use.)

<Apply L'Hôpital's Rule>

$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^x} = \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty}\frac{1}{2\sqrt{x}\,e^x} = 0$$

If rewriting a limit in one of the forms $\frac{0}{0}$, $\frac{\infty}{\infty}$ does not seem to work, try ..another form

For example, the previous problem could have been Rewritten as $\lim_{x\to\infty} e^{-x}\sqrt{x} = \lim_{x\to\infty}\frac{e^{-x}}{x^{-1/2}}$ <which yields $\frac{0}{0}$>

The indeterminate forms $1^\infty, \infty^0, 0^0$ arise from limits of functions that have variable bases and variables exponents. When you find these, … use logarithmic differentiation to find the derivative

Example:

Evaluate $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ Because direct substitution yields 1^∞,

Solution: First, set limit equal to y:

$$y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Take natural log of both sides:

$$\ln y = \ln\left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right] = \lim_{x\to\infty}\left[x\ln\left(1+\frac{1}{x}\right)\right] = \lim_{x\to\infty}\left(\frac{\ln[1+(1/x)]}{1/x}\right)$$

$$= \lim_{x\to\infty}\frac{1}{1+(1/x)} \overset{\text{we can see}}{=} 1$$

Now, $\ln y = 1 \rightarrow e^{\ln y} = e^1$
$y = e \quad \therefore \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Example:

Evaluate $\lim_{x \to \cancel{\infty} 0^+} (\sin x)^x$ Because direct substition yields 0^0

Solution: First, set limit equal to y:

$$y = \lim_{x \to 0^+} (\sin x)^x$$

Take ln of both sides:

$$\ln y = \ln\left[\lim_{x \to 0^+} (\sin x)^x\right] = \lim_{x \to 0^+} \left[\ln(\sin x)^x\right] = \lim_{x \to 0^+} \left[x \ln(\sin x)\right]$$

$$= \lim_{x \to 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x \to 0^+} \frac{\cot x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{\tan x}$$

<Apply L'Hopital's Rule>

$$= \lim_{x \to 0^+} \frac{-2x}{\sec^2 x} = 0$$

Now, because $\ln y = 0 \xrightarrow{\text{we conclude}} y = e^0 = 1$ $\therefore$ $\lim_{x \to 0^+} (\sin x)^x = 1$

Example:

Evaluate $\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right)$ Because direct substitution yields $\infty - \infty$ we need to rewrite to find a way to generate $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Solution: First, combine 2 fractions:

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \to 1^+} \left[\frac{x-1-\ln x}{(x-1)\ln x} \right] \xrightarrow{\text{this will yield}} \frac{0}{0}$$

Now, we can apply L'Hôpital's Rule:

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \to 1^+} \left[\frac{1-0-(1/x)}{(x-1)(1/x)+\ln x} \right] = \lim_{x \to 1^+} \left(\frac{x-1}{x-1+x\ln x} \right) \to \frac{0}{0}$$

L'Hôpital's again:

$$= \lim_{x \to 1^+} \left[\frac{1}{1-0+x(1/x)+\ln x} \right] = \frac{1}{2}$$

HW pp.530-531: 5-15:odd; #18, #19, #23, #24, #25, #26, #29, #30